

STABILITY CONDITIONS VIA SPHERICAL OBJECTS

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ABSTRACT. An object in the bounded derived category $D^b(X)$ of coherent sheaves on a complex projective K3 surface X is spherical if it is rigid and simple. Although spherical objects form only a discrete set in the moduli stack of complexes, they determine much of the structure of X and $D^b(X)$. Here we show that a stability condition on $D^b(X)$ is determined by the stability of spherical objects.

Consider the bounded derived category $D^b(X)$ of the abelian category $\text{Coh}(X)$ of coherent sheaves on a complex projective K3 surface X . An object A in $D^b(X)$, i.e. a bounded complex of coherent sheaves on X , is called *spherical* if

$$\text{Ext}^*(A, A) \simeq H^*(S^2, \mathbb{C}).$$

Spherical objects play a central role in the theory of K3 surfaces. In the classical theory they occur as (-2) -curves, i.e. smooth rational curves $\mathbb{P}^1 \simeq C \subset X$, and indeed the structure sheaf \mathcal{O}_C of a (-2) -curve C is a spherical object in $D^b(X)$. Other examples of spherical objects in $D^b(X)$ are provided by line bundles L or, more generally, rigid stable vector bundles.

More recently, spherical objects A and their associated spherical twists T_A have been used to give a conjectural description of the group of all exact linear autoequivalences $\text{Aut}(D^b(X))$ (see [4]). In this language, the reflection s_C classically associated to a (-2) -curve C and used to generate the Weyl group W_X of a K3 surface, can be reinterpreted as the cohomological action of the spherical twist $T_{\mathcal{O}_C(-1)}$. Spherical objects also seem to play a role in the study of Chow groups and the arithmetic of K3 surfaces (see [9, 10]).

Clearly, point sheaves $k(x)$, which are semirigid objects in $D^b(X)$, determine the geometry of X completely. But it seems that the much smaller discrete set $\mathcal{S} \subset D^b(X)$ of all spherical objects carries essentially the same information.

The purpose of the present article is to stress this point further by showing that a stability condition on the derived category $D^b(X)$ is determined by the phases of only spherical objects.

Stability conditions as introduced by Bridgeland in [3] have been studied intensively for K3 categories. In [4] Bridgeland studies a distinguished connected component Σ of the space $\text{Stab}(X) := \text{Stab}(D^b(X))$ (as before, X a projective K3 surface) of all stability conditions and conjectures that Σ is simply-connected and preserved by the group of exact autoequivalences $\text{Aut}(D^b(X))$. For generic non-projective K3 surfaces $\text{Stab}(X)$ was completely described in [7].

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The case of local K3 surfaces is more accessible. More precisely, for the minimal resolution $\pi : X \rightarrow \mathbb{C}^2/G$ of a Kleinian singularity one can consider K3 categories $\mathcal{D} \subset \hat{\mathcal{D}} \subset \mathrm{D}^b(X)$ of complexes supported on the exceptional divisor (resp. with vanishing $R\pi_*$). The spaces $\mathrm{Stab}(\mathcal{D})$ and $\mathrm{Stab}(\hat{\mathcal{D}})$ are studied in detail in [11, 15] for A_n -singularities and in [2, 5] in general. Roughly, the analogue of Bridgeland's conjecture, originally formulated for projective K3 surfaces, are known to hold in the local situation.

There are at least two reasons why stability conditions in the local situation can be understood almost completely while the case of projective K3 surfaces still eludes us. Firstly, the group of autoequivalences of $\mathrm{D}^b(X)$ for a projective K3 surface X is much more complex than 'just' a braid group. Indeed, $\mathrm{D}^b(X)$ can host many different A_n -configurations of spherical objects at a time, which might be interlinked in a complicated manner. Secondly, in the local case the categories under consideration are generated by spherical objects and, in particular, their Grothendieck groups are of finite rank. A priori, the structure of $\mathrm{D}^b(X)$ for a projective K3 surface seems more complicated due to the many objects not generated by spherical objects.

The goal of this paper is to show that also for a projective K3 surface X the space of stability conditions on $\mathrm{D}^b(X)$ can be studied purely in terms of a configuration of spherical objects, in other words in terms of a category that is spanned by spherical objects. In some sense this is meant to bridge the gap between the existing work in the local and in the global setting, but whether it can be useful to prove Bridgeland's conjecture remains to be seen.

The main result of the paper (see Theorem 3.1) is concerned with two stability conditions $\sigma = (\mathcal{P}, Z)$ and $\sigma' = (\mathcal{P}', Z')$ in the distinguished connected component Σ of the space of all stability conditions $\mathrm{Stab}(X)$.

Theorem 0.1. *Assume $Z = Z'$. Then*

$$\sigma = \sigma'$$

if and only if for all spherical objects A :

$$A \text{ is } \sigma\text{-stable of phase } \varphi \text{ if and only if } A \text{ is } \sigma'\text{-stable of phase } \varphi.$$

The result can be reformulated in terms of a new metric on $\mathrm{Stab}(X)$, only taking spherical objects into account, which by the theorem turns out to be equivalent to the one defined by Bridgeland in [3] (see Corollary 4.5).

This point of view is the motivation for the following construction. Consider the full triangulated subcategory $\mathcal{S}^* \subset \mathrm{D}^b(X)$ generated by \mathcal{S} . Note that in generating \mathcal{S}^* we do not allow taking direct summands (see [10] for details). Then \mathcal{S}^* is dense in $\mathrm{D}^b(X)$ and its Grothendieck group $K(\mathcal{S}^*) \subset K(X)$ equals $N(X) = \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$ (under the additional but presumably superfluous assumption $\rho(X) \geq 2$).

The triangulated category \mathcal{S}^* does not carry a bounded t-structure and, therefore, no stability condition. But considering a weaker notion of stability conditions one can introduce $\mathrm{Stab}(\mathcal{S}) = \mathrm{Stab}(\mathcal{S}^*)$ which as in [3] is endowed with a natural (generalized) metric (see Section 4.3). The restriction of a stability condition on $\mathrm{D}^b(X)$ to $\mathcal{S}^* \subset \mathrm{D}^b(X)$ is then well-defined, i.e. there exists a continuous map $\mathrm{Stab}(X) \rightarrow \mathrm{Stab}(\mathcal{S})$.

As a consequence of Theorem 0.1 one obtains the following (see Corollary 4.9)

Corollary 0.2. *On the distinguished component the restriction yields an embedding*

$$\Sigma \hookrightarrow \mathrm{Stab}(\mathcal{S})$$

which identifies the natural metric on $\mathrm{Stab}(\mathcal{S})$ with the spherical metric $d_{\mathcal{S}}$ on Σ .

Note that for \mathcal{S}^* there is no difference between the Grothendieck group of \mathcal{S}^* and its quotient by numerical equivalence $\sim_{\mathcal{S}}$. Thus $K(\mathcal{S}^*)/\sim_{\mathcal{S}} \simeq K(X)/\sim = N(X)$ and, therefore, maximal components of $\mathrm{Stab}(\mathcal{S})$ and $\mathrm{Stab}(X)$ are modeled locally over the same linear space.

Here is an outline of the paper. Section 1 contains the basic definitions and results on stability conditions on $D^b(X)$ and explains some useful techniques (Lemma 1.3, 1.4) to study the heart of a standard stability condition. In Section 2 we recall that stable factors of a spherical objects are again spherical and study spherical objects in the heart of a standard stability condition. Section 3 contains the proof of the main theorem. It is first proved for the case that one of the stability conditions is standard. The generic case can be reduced to this by applying autoequivalences, but the case of stability conditions in the boundary of the set of standard stability conditions is more complicated. The result can be rephrased in terms of the spherical metric, which is explained in Section 4. The last part could be read together with [10] which discusses the category \mathcal{S}^* from a different angle and in more detail. The appendix collects a few observations on the groups $\mathrm{Aut}(D^b(X))$ and $\mathrm{Aut}(X)$.

1. GENERAL REMARKS ON STABILITY CONDITIONS

1.1. Recall that a stability condition $\sigma = (\mathcal{P}, Z)$ on a triangulated category \mathcal{D} as introduced by Bridgeland in [3] consists of a slicing \mathcal{P} and a stability function Z .

The *slicing* \mathcal{P} of σ is given by full abelian subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$, $\phi \in \mathbb{R}$. The slicing has two properties:

$$\mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0 \text{ for } \phi_1 > \phi_2 \text{ and } \mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1).$$

Objects in $\mathcal{P}(\phi)$ are the *semistable objects of phase* ϕ . Let $\mathcal{P}(\phi)^s \subset \mathcal{P}(\phi)$ denote the subcategory of all *stable* objects $E \in \mathcal{P}(\phi)$, i.e. objects $E \in \mathcal{P}(\phi)$ not containing any proper non-trivial subobject in $\mathcal{P}(\phi)$.

The *stability function* is a linear function $Z : K(\mathcal{D}) \rightarrow \mathbb{C}$ such that $Z(E) \in \exp(i\pi\phi)\mathbb{R}_{>0}$ for all $0 \neq E \in \mathcal{P}(\phi)$.

We shall only consider locally finite numerical stability conditions. The latter means that the stability function factors as $Z : K(\mathcal{D}) \rightarrow \Lambda \rightarrow \mathbb{C}$ with $\Lambda := K(\mathcal{D})/\sim$ the numerical Grothendieck group of \mathcal{D} .

The finiteness of the stability condition is a technical assumption that in [3] enters the discussion of the topology on the space of stability conditions. Here, finiteness will explicitly only be used to ensure that the abelian categories $\mathcal{P}(\phi)$ are of finite length, i.e. any semistable object has a finite filtrations with stable quotients.

In particular, for any stability condition $\sigma = (\mathcal{P}, Z)$ and any object $0 \neq E \in \mathcal{D}$ there exists a σ -stable decomposition, i.e. a diagram of exact triangles

$$(1.1) \quad \begin{array}{ccccccc} & F_1 & \xrightarrow{\quad} & F_2 & \longrightarrow & \cdots & \longrightarrow & F_{m-1} & \xrightarrow{\quad} & F_m \equiv E \\ & \parallel & & \swarrow & & & & \swarrow & & \swarrow \\ A_1 & & & [1] & & & & [1] & & A_m \end{array}$$

with $A_i \in \mathcal{P}(\phi_i)$ and such that $\phi_1 \geq \dots \geq \phi_m$. The minimal and maximal phases of E are defined as $\phi^-(E) := \phi_m$ resp. $\phi^+(E) := \phi_1$; they are uniquely determined. The A_i are called the σ -stable factors of E and they are unique up to permutation among those of the same phase. Note that as in the classical case the two morphisms $A_1 \rightarrow E$ and $E \rightarrow A_m$ are always non-trivial.

Requiring strict inequalities leads to the Harder–Narasimhan (or σ -semistable) decomposition by σ -semistable factors. This decomposition is unique.

1.2. Let $\text{Stab}(\mathcal{D})$ be the space of all (locally finite and numerical) stability conditions on \mathcal{D} . In [3] Bridgeland uses a generalized metric to define a topology on $\text{Stab}(\mathcal{D})$. The distance between two slicings \mathcal{P} and \mathcal{P}' is measured by

$$f(\mathcal{P}, \mathcal{P}') := \sup_{0 \neq E \in \mathcal{D}} \{|\phi^+(E) - \phi'^+(E)|, |\phi^-(E) - \phi'^-(E)|\}$$

where ϕ^\pm and ϕ'^\pm are the minimal (resp. maximal) phases with respect to \mathcal{P} resp. \mathcal{P}' .

The generalized metric $d(\sigma, \sigma')$ between two stability conditions $\sigma = (\mathcal{P}, Z), \sigma' = (\mathcal{P}', Z') \in \text{Stab}(X)$ combines $f(\mathcal{P}, \mathcal{P}')$ with a distance function for $\sum |Z(A_i)|$ and $\sum |Z'(A'_i)|$ for the respective stable decompositions of all $E \in \mathcal{D}$. But on each connected component of $\text{Stab}(\mathcal{D})$ it is in fact equivalent to the product metric

$$d(\sigma, \sigma') := \max\{f(\mathcal{P}, \mathcal{P}'), |Z - Z'|\}.$$

As we will restrict to a connected component from the outset, we shall work with this simpler distance function. Note that due to the definition of $f(\mathcal{P}, \mathcal{P}')$, taking into account all objects $E \in \text{D}^b(X)$, the distance between two stability function is difficult to compute explicitly.

1.3. We shall now specialize to the case that \mathcal{D} is the bounded derived category $\text{D}^b(X) := \text{D}^b(\text{Coh}(X))$ of the abelian category of coherent sheaves on a complex projective K3 surface. We shall write $\text{Stab}(X)$ for $\text{Stab}(\text{D}^b(X))$.

Stability conditions on higher-dimensional varieties are difficult to construct. On K3 surfaces, Bridgeland constructs in [4] explicit examples of stability conditions as follows. Let $\omega \in \text{NS}(X)_\mathbb{R}$ be an ample class and let $B \in \text{NS}(X)_\mathbb{R}$ be arbitrary. Consider the linear function

$$E \mapsto Z(E) = \langle \exp(B + i\omega), v(E) \rangle.$$

Here, $v(E) = \text{ch}(E)\sqrt{\text{td}(X)} \in N(X) \subset H^*(X, \mathbb{Z})$ is the Mukai vector of E and $\langle \cdot, \cdot \rangle$ is the Mukai pairing.

Under the additional condition that $Z(E) \notin \mathbb{R}_{<0}$ for all spherical sheaves (which holds whenever $\omega^2 > 2$), the function Z has the Harder–Narasimhan property on the abelian category $\mathcal{A}(\exp(B + i\omega))$ which is defined as follows (see [4, Sect. 7]).

An object $E \in \mathcal{D}^b(X)$ is contained in $\mathcal{A}(\exp(B + i\omega))$ if and only if there exists an exact triangle

$$(1.2) \quad \mathcal{H}^{-1}[1] \longrightarrow E \longrightarrow \mathcal{H}^0$$

with coherent sheaves $\mathcal{H}^{-1}, \mathcal{H}^0$ satisfying:

- i) \mathcal{H}^{-1} is zero or torsion free with $\mu_{\max} \leq (B, \omega)$.
- ii) \mathcal{H}^0 is torsion or $\mu_{\min} > (B, \omega)$.

The category $\mathcal{A}(\exp(B + i\omega))$ is the heart of a t-structure that is obtained by tilting the standard t-structure with respect to the torsion theory described by i) and ii). This defines a stability condition σ depending on $B + i\omega$ whose heart, i.e. the abelian category $\mathcal{P}(0, 1]$ of all objects with $\phi^\pm \in (0, 1]$, is precisely $\mathcal{A}(\exp(B + i\omega))$. We will refer to these stability conditions as *standard stability conditions*.

Standard stability conditions form a subset $V(X) \subset \text{Stab}(X)$ which via the period map $\sigma = (\mathcal{P}, Z) \mapsto Z$ and the Mukai pairing, can be identified with a subset of $N(X)_\mathbb{C}$, where $N(X) := H^0 \oplus \text{NS}(X) \oplus H^4$ is the algebraic part of $H^*(X, \mathbb{Z})$. The set of standard stability conditions $V(X)$ can intrinsically be described as follows, see [4, Prop. 10.3].

Proposition 1.1. (Bridgeland) *Suppose σ is a stability condition with respect to which for all points $x \in X$ the skyscraper sheaf $k(x)$ is σ -stable of phase one. Then $\sigma \in V(X)$. \square*

The natural $\widetilde{\text{Gl}}^+(2, \mathbb{R})$ -action on $\text{Stab}(X)$ can be used to describe the set $U(X)$ of all stability conditions with respect to which all point sheaves $k(x)$ are stable of the same phase. Indeed, $U(X) = V(X) \cdot \widetilde{\text{Gl}}^+(2, \mathbb{R})$ which can also be viewed as a $\widetilde{\text{Gl}}^+(2, \mathbb{R})$ -bundle over $V(X)$.

The connected component of $\text{Stab}(X)$ containing $V(X)$ will be denoted Σ . For Σ one has the following description due to Bridgeland [4]. Consider the open set $\mathcal{P}(X)$ of all classes in $N(X)_\mathbb{C}$ whose real and imaginary part span a positive plane and let $\mathcal{P}^+(X)$ be the connected component of $\mathcal{P}(X)$ that contains all $\exp(B + i\omega)$ with ample ω . Then one defines $\mathcal{P}_0^+(X)$ as the open subset $\mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta} \delta^\perp$, where $\Delta \subset N(X)$ is the set of (-2) -classes.

Proposition 1.2. (Bridgeland) *The period map $\sigma = (\mathcal{P}, Z) \mapsto Z$ yields a covering map*

$$\Sigma \longrightarrow \mathcal{P}_0^+(X).$$

The group of deck transformation $\text{Gal}(\Sigma/\mathcal{P}_0^+(X))$ is naturally identified with the group of all derived equivalences Φ preserving Σ and acting trivially on $H^(X, \mathbb{Z})$.*

1.4. For the convenience of the reader we provide the following list of mostly rather obvious facts on coherent sheaves on a projective K3 surface X . We fix an ample line bundle $\mathcal{O}(1)$.

Lemma 1.3. *i) If F is a locally free sheaf, then $\text{Ext}^1(\mathcal{O}(n), F) \simeq H^1(X, F^*(n))^* = 0$ for $n \gg 0$.
ii) If $F \in \text{Coh}(X)$ and $\text{Hom}(\mathcal{O}(n), F) \neq 0$ for $n \gg 0$, then F contains a non-trivial subsheaf $G \subset F$ with zero-dimensional support.*

iii) If $F \in \text{Coh}(X)$ is simple, i.e. $\text{End}(F) \simeq \mathbb{C}$, then F does not contain a non-trivial proper subsheaf $0 \neq G \subsetneq F$ with zero-dimensional support.

iv) If $F \in \text{Coh}(X)$ is rigid and torsion free, then F is locally free.

Proof. Serre duality and Serre vanishing imply i). In order to prove ii), one can argue as follows. A generic section $t \in H^0(X, \mathcal{O}(n))$, $n \gg 0$, induces naturally an injection $F(-n) \hookrightarrow F$. Thus $h^0(F(-n)) \leq h^0(F)$. If indeed $H^0(X, F(-n)) = \text{Hom}(\mathcal{O}(n), F) \neq 0$ for $n \gg 0$, then we may assume that $h^0(F(-n)) = h^0(F) \neq 0$ for all $n > 0$ (pass to $F(-n_0)$, $n_0 \gg 0$, if necessary). Then choose $0 \neq s \in H^0(X, F)$ and write it as $s : \mathcal{O}_X \rightarrow \mathcal{O}_Z \hookrightarrow F$ for some non-empty subscheme $Z \subset X$.

For generic $C \in |\mathcal{O}(n)|$, $n \gg 0$ one has exact sequences $0 \rightarrow \mathcal{O}_Z(-n) \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{Z \cap C} \rightarrow 0$ and $0 \rightarrow F(-n) \rightarrow F \rightarrow F_C \rightarrow 0$ with $\mathcal{O}_{Z \cap C} \hookrightarrow F_C$ and hence $H^0(\mathcal{O}_{Z \cap C}) \subset H^0(F_C)$. Since $H^0(X, F(-n)) = H^0(X, F)$, the section $s \in H^0(\mathcal{O}_Z) \subset H^0(F)$ is contained in $H^0(\mathcal{O}_Z(-n))$. But $H^0(\mathcal{O}_Z(-n)) = 0$ for $n > 0$ except for $\dim Z = 0$.

For iii) consider the non-trivial quotient $F' := F/G$. If G has zero-dimensional support and $x \in \text{Supp } G$, then $k(x) \subset G$. If also $x \in \text{Supp } F'$, then there exists a surjection $F' \twoheadrightarrow k(x)$ which by composition with $F \twoheadrightarrow F'$ and $k(x) \hookrightarrow G \subset F$ yields an endomorphism of F which is not of the form $\lambda \cdot \text{id}$. If F' and G have disjoint support, then $F \simeq F' \oplus G$ which is clearly not simple.

For iv) consider the reflexive hull F^{**} of F . The quotient S of $F \subset F^{**}$ is concentrated in dimension zero and the natural surjection $F^{**} \twoheadrightarrow S$ can be deformed such that S changes its support. Taking kernels yields a deformation of F which really is non-trivial as the support of its singular part deforms. This contradicts the assumption that F is rigid. A more explicit dimension count is expressed in [13, Prop. 2.14]. \square

From these easy facts one can deduce useful information on the heart of a standard stability condition. Let $\omega \in \text{NS}(X)_{\mathbb{R}}$ be an ample class, $B \in \text{NS}(X)_{\mathbb{R}}$, and let σ be the standard stability condition with stability function $Z = \langle \exp(B + i\omega), \cdot \rangle$ and heart $\mathcal{A} := \mathcal{A}(B + i\omega) = \mathcal{P}(0, 1]$ (see Section 1.3).

Lemma 1.4. *If $E \in \mathcal{A}$, then $\text{Hom}(E, \mathcal{O}(-n)[k]) = 0$ for $n \gg 0$ and $k \leq 1$.*

Proof. By construction, $\mathcal{O}(-n) \in \mathcal{A}[-1]$ for $n \gg 0$ or, more precisely, for $-n \leq (B.\omega)/(\mathcal{O}(1).\omega)$. Hence, $\text{Hom}(\mathcal{A}, \mathcal{O}(-n)[k]) = 0$ for $k \leq 0$. For $k = 1$ use Serre duality to write

$$\text{Hom}(E, \mathcal{O}(-n)[1]) = \text{Ext}^1(E, \mathcal{O}(-n)) \simeq \text{Ext}^1(\mathcal{O}(-n), E)^*.$$

Then apply $\text{Hom}(\mathcal{O}(-n), \cdot)$ to (1.2) which yields the exact sequence

$$\text{Ext}^2(\mathcal{O}(-n), \mathcal{H}^{-1}) \rightarrow \text{Ext}^1(\mathcal{O}(-n), E) \rightarrow \text{Ext}^1(\mathcal{O}(-n), \mathcal{H}^0).$$

Then for $n \gg 0$ Serre vanishing yields $\text{Ext}^2(\mathcal{O}(-n), \mathcal{H}^{-1}) = H^2(X, \mathcal{H}^{-1}(n)) = 0$ and similarly $\text{Ext}^1(\mathcal{O}(-n), \mathcal{H}^0) = H^1(X, \mathcal{H}^0(n)) = 0$. \square

A similar ‘dual’ statement for spherical objects will be proved in Lemma 2.6.

2. SPHERICAL OBJECTS

2.1. Let us recall the definition of a spherical object. We shall work with a *K3 category* \mathcal{D} which later will be $\text{D}^b(X)$, the bounded derived category of coherent sheaves on a K3 surface X . Recall that a K3 category is a linear triangulated category of finite type with the shift [2] defining a Serre functor.

Definition 2.1. An object $A \in \mathcal{D}$ is called *spherical* if

$$\mathrm{Ext}^*(A, A) \simeq H^*(S^2, \mathbb{C}).$$

By $\mathcal{S} \subset \mathrm{Ob}(\mathcal{D})$ we denote the collection of all spherical objects in \mathcal{D} .

Thus $A \in \mathcal{S}$ if and only if A is simple (i.e. $\mathrm{End}(A) \simeq \mathbb{C}$), rigid (i.e. $\mathrm{Ext}^1(A, A) = 0$), and $\mathrm{Ext}^i(A, A) = 0$ for $i < 0$. The easiest examples are provided by line bundles on a K3 surface X viewed as objects in the K3 category $\mathrm{D}^b(X)$.

To shorten the notation we will sometimes write $\mathrm{ext}^i(A, B) = \dim \mathrm{Ext}^i(A, B)$. The following results go back to Mukai, see e.g. [13, Cor. 2.8]. In this form they can be found in [7, Lem. 2.7, Prop. 2.9] (see also [4, Lem. 12.2]).

Lemma 2.2. *Consider in the K3 category \mathcal{D} an exact triangle*

$$A \xrightarrow{i} E \xrightarrow{j} B \xrightarrow{\delta} A[1]$$

such that

$$\mathrm{Ext}^r(A, B) = \mathrm{Ext}^s(B, B) = 0 \text{ for } r \leq 0 \text{ and } s < 0.$$

Then

$$\mathrm{ext}^1(A, A) + \mathrm{ext}^1(B, B) \leq \mathrm{ext}^1(E, E).$$

The following two consequences hold true for arbitrary slicings, no stability function is needed.

Corollary 2.3. *Let σ be a stability condition on \mathcal{D} and $A \in \mathcal{S}$. If A_1, \dots, A_k are the σ -stable factors of A (cf. Section 1.1), then $A_1, \dots, A_k \in \mathcal{S}$. \square*

An object $E \in \mathcal{D}$ is called *semirigid* if $\mathrm{Ext}^1(E, E)$ is two-dimensional. If $x \in X$ is a closed point of a K3 surface X , then $k(x)$ is a semirigid object in $\mathrm{D}^b(X)$.

Corollary 2.4. *Let σ be a stability condition on a K3 category \mathcal{D} and let E be a semirigid object. Then the σ -stable factors E_1, \dots, E_k of E are spherical or semirigid. In fact, at most one E_i can be semirigid. \square*

2.2. Consider two stability conditions σ and σ' on the K3 category \mathcal{D} . The proof of the following result only uses the underlying slicings, \mathcal{P} resp. \mathcal{P}' , and the property that all $\mathcal{P}(\phi)$ and $\mathcal{P}'(\phi)$ are abelian.

Proposition 2.5. *The following conditions are equivalent:*

- i) *For all $\phi \in \mathbb{R}$ one has $\mathcal{P}(\phi)^s \cap \mathcal{S} = \mathcal{P}'(\phi)^s \cap \mathcal{S}$.*
- ii) *For all $\phi \in \mathbb{R}$ one has $\mathcal{P}(\phi) \cap \mathcal{S} = \mathcal{P}'(\phi) \cap \mathcal{S}$.*
- iii) *For all $A \in \mathcal{S}$ one has $\phi^\pm(A) = \phi'^\pm(A)$.*

Proof. Assume iii). An object E is σ -semistable of phase ϕ if and only if $\phi^+(E) = \phi^-(E) = \phi$. But for $A \in \mathcal{S}$ this is, assuming iii), equivalent to $\phi'^+(A) = \phi'^-(A) = \phi$. Hence such an A is also σ' -semistable of the same phase ϕ . Thus, ii) holds.

Assume ii). If $A \in \mathcal{S}$ is σ -stable of phase ϕ , then A is in particular σ -semistable of phase ϕ and hence by ii) also σ' -semistable of phase ϕ . If A is not σ' -stable, then there exists a minimal proper subobject $A' \subset A$ in the abelian category $\mathcal{P}'(\phi)$. Then A' is σ' -stable and as

a stable factor of a spherical object, A' is also spherical (cf. Corollary 2.3). Hence by ii), A' is also σ -semistable of phase ϕ . One would now like to argue that then the inclusion $A' \subset A$ in $\mathcal{P}'(\phi)$ must be an isomorphism because A was σ -stable. However a priori we do not know that $A' \rightarrow A$ is still an injection in $\mathcal{P}(\phi)$. But since $A \in \mathcal{P}(\phi)$ is σ -stable, $A' \in \mathcal{P}(\phi)$, and $A' \rightarrow A$ is non-trivial, A is a σ -stable factor of A' and $A' \rightarrow A$ is a surjection in $\mathcal{P}(\phi)$. The σ -stable factors of its kernel in $\mathcal{P}(\phi)$ are σ -stable factors of the spherical A' and hence also spherical. Thus the short exact sequence $0 \rightarrow \text{Ker} \rightarrow A' \rightarrow A \rightarrow 0$ in $\mathcal{P}(\phi)$ is also a short exact sequence in $\mathcal{P}'(\phi)$, but as A' was a subobject of A in $\mathcal{P}'(\phi)$ this shows $\text{Ker} = 0$. Hence $A' \simeq A$ and thus A is σ' -stable of phase ϕ . This shows i).

Assume i). Consider a σ -stable filtration $F_1 \rightarrow \dots \rightarrow F_n = A$ with σ -stable factors A_i of phase ϕ_i . Since $A \in \mathcal{S}$, all $A_i \in \mathcal{S}$. Hence, all A_i are by i) also σ' -stable of phase ϕ_i . In particular, the given filtration is also a stable filtration with respect to σ' . But then $\phi^+(A) = \phi_1 = \phi'^+(A)$ and $\phi^-(A) = \phi_n = \phi'^-(A)$. This shows iii). \square

2.3. In analogy to Lemma 1.4 one has the following result for spherical objects in the bounded derived category $\text{D}^b(X)$ of a complex projective K3 surface X . As before, \mathcal{A} is the heart of a standard stability condition with stability function $Z = \langle \exp(B + i\omega), \cdot \rangle$.

Lemma 2.6. *If $A \in \mathcal{A}$ is spherical, then $\text{Hom}(\mathcal{O}(n), A[k]) = 0$ for all $k \leq 0$ and $n \gg 0$.*

Proof. By stability, $\text{Hom}(\mathcal{A}, \mathcal{A}[k]) = 0$ for $k < 0$. Since $\mathcal{O}(n) \in \mathcal{A}$ for $n \gg 0$, or more precisely for $n > (B.\omega)/(\mathcal{O}(1).\omega)$, this proves the vanishing for negative k . To prove the vanishing for $k = 0$ apply $\text{Hom}(\mathcal{O}(n), \cdot)$ to (1.2) for A which yields the exact sequence

$$\text{Ext}^1(\mathcal{O}(n), \mathcal{H}^{-1}) \rightarrow \text{Hom}(\mathcal{O}(n), A) \rightarrow \text{Hom}(\mathcal{O}(n), \mathcal{H}^0).$$

As A is spherical and $\text{Hom}(\mathcal{H}^{-1}[1], \mathcal{H}^0) = 0$, Lemma 2.2 shows that \mathcal{H}^{-1} and \mathcal{H}^0 are both rigid sheaves. Thus \mathcal{H}^{-1} is a rigid torsion free sheaf and therefore locally free (see Lemma 1.3, iv)). By Lemma 1.3, i) one finds $\text{Ext}^1(\mathcal{O}(n), \mathcal{H}^{-1}) = 0$ for $n \gg 0$. Thus, if $\text{Hom}(\mathcal{O}(n), A) \neq 0$ for $n \gg 0$, then $\text{Hom}(\mathcal{O}(n), \mathcal{H}^0) \neq 0$ for $n \gg 0$. By Lemma 1.3, ii), this means that the zero-dimensional part $G := T_0(\mathcal{H}^0) \subset \mathcal{H}^0$ of \mathcal{H}^0 is non-trivial. If \mathcal{H}^0 is not only rigid but in fact spherical, then Lemma 1.3, iii) would show that \mathcal{H}^0 is zero-dimensional and in fact $\mathcal{H}^0 \simeq k(x)$. Clearly, the latter would contradict rigidity of \mathcal{H}^0 . If \mathcal{H}^0 is rigid but not simple, one can argue as follows. Note that $\text{Ext}^i(G, \mathcal{H}^0/G) = 0$ for $i \leq 0$ and $\text{Ext}^i(\mathcal{H}^0/G, \mathcal{H}^0/G) = 0$ for $i < 0$. Then by Lemma 2.2 one finds $\text{ext}^1(G, G) + \text{ext}^1(\mathcal{H}^0/G, \mathcal{H}^0/G) \leq \text{ext}^1(\mathcal{H}^0, \mathcal{H}^0) = 0$, but clearly the zero-dimensional sheaf G deforms and hence $\text{Ext}^1(G, G) \neq 0$ which yields a contradiction. \square

3. STABILITY CONDITIONS VIA SPHERICAL OBJECTS

Let X be a complex projective K3 surface and $\Sigma \subset \text{Stab}(X)$ the distinguished connected component of the space of locally finite numerical stability conditions on $\text{D}^b(X)$ (see [4]).

This section is entirely devoted to the proof of the following

Theorem 3.1. *Suppose $\sigma = (\mathcal{P}, Z), \sigma' = (\mathcal{P}', Z')$ are stability conditions in Σ . Then*

$$\sigma = \sigma'$$

(3.1) *A is σ -semistable of phase φ if and only if A is σ' -semistable of phase φ .*

The proof proceeds in three steps. We shall first assume that σ is a standard stability condition (Section 3.1) and then reduce to this case by applying autoequivalences. The case that σ can only be transformed into a stability condition that is a limit of standard stability conditions will be dealt with in Section 3.3

(3.2) A is σ -stable of phase φ if and only if A is σ' -stable of phase φ .

To shorten the notation we shall denote the heart $\mathcal{A}(B + i\omega) = \mathcal{P}(0, 1]$ of σ simply by \mathcal{A} .

Proof. Pick a line bundle L with $(L.\omega) > (B.\omega)$. Then $L \in \mathcal{A}$ by definition of $\mathcal{A} = \mathcal{A}(B + i\omega)$. The line bundle L is a spherical object and by Corollary 2.3 all σ -stable factors L_i of L are spherical as well. Since $L \in \mathcal{A}$, their phases satisfy $\phi(L_i) \in (0, 1]$.

By our assumption on σ' (see (3.2)), the L_i are then also σ' -stable of phase $\phi'(L_i) = \phi(L_i) \in (0, 1]$. Clearly, any line bundle L admits a non-trivial morphism $L \rightarrow k(x)$ and hence at least one of the σ -stable factors L_i admits a non-trivial morphism $L_i \rightarrow k(x)$. Since we assume $k(x)$ to be σ' -stable, its σ' -phase is well defined and thus satisfies $0 < \phi'(L_i) \leq \phi'(k(x))$. On the other hand, by Serre duality, $\text{Hom}(L_i, k(x)) \neq 0$ implies $\text{Hom}(k(x), L_i[2]) \neq 0$. The latter yields $\phi'(k(x)) \leq \phi'(L_i[2]) \leq 3$. Moreover, $\phi'(k(x)) = \phi'(L_i) = 3$ can only occur if the two σ' -stable objects $k(x)$ and $L_i[2]$ are isomorphic, which is absurd as one is semirigid and the other is spherical. Thus, $\phi'(k(x)) \in (0, 3)$. As $Z = Z'$ and $Z(k(x)) = -1$, this readily shows $\phi'(k(x)) = 1$. \square

Suppose $k(x)$ is not σ' -stable. Then there exists a σ' -stable decomposition, i.e. a diagram

$$\begin{array}{ccccccc}
 & F_1 & \longrightarrow & F_2 & \longrightarrow & \cdots & \longrightarrow & F_{m-1} & \longrightarrow & F_m & \equiv & k(x) \\
 & \parallel & & \swarrow & & & & \swarrow & & \swarrow & & \\
 A_1 & & & [1] & & & & [1] & & & & \\
 & & & A_2 & & & & & & A_m & &
 \end{array}$$

where the A_i are σ' -stable with $\phi'(A_1) \geq \dots \geq \phi'(A_m)$ and $m > 1$. By Corollary 2.4 at most one A_i is not spherical and if there is one, it is semirigid.

i) Suppose A_1 and A_m are both spherical. Then by (3.2), both are also σ -stable and for their phases one has $\phi(A_1) = \phi'(A_1)$ and $\phi(A_m) = \phi'(A_m)$. Since $A_1 = F_1 \rightarrow k(x)$ is not trivial, $\phi(A_1) \leq \phi(k(x)) = 1$ and equality would imply $A_1 = k(x)$ which can be excluded as in the proof of Lemma 3.2. Similarly, $k(x) = F_m \rightarrow A_m$ is not trivial and hence $1 = \phi(k(x)) \leq \phi(A_m)$. This yields the contradiction $1 > \phi(A_1) = \phi'(A_1) \geq \phi'(A_m) = \phi(A_m) \geq 1$.

ii) Suppose A_m is semirigid. Then A_1, \dots, A_{m-1} are spherical and hence also σ -stable with phases $\phi(A_i) = \phi'(A_i)$, $i = 1, \dots, m-1$. As above, $\text{Hom}(A_1, k(x)) \neq 0$ implies $1 > \phi(A_1) = \phi'(A_1) \geq \dots \geq \phi(A_{m-1}) = \phi'(A_{m-1})$. Thus $A_i \in \mathcal{A}[k_i]$, $i = 1, \dots, m-1$, with $k_i \leq 0$. Then Lemma 2.6 shows $\text{Hom}(\mathcal{O}(n), A_i) = 0$ for $i = 1, \dots, m-1$ and $n \gg 0$. Since $\text{Hom}(\mathcal{O}(n), k(x)) \neq 0$ for all n , we find $\text{Hom}(\mathcal{O}(n), A_m) \neq 0$ for $n \gg 0$.

Clearly, $\mathcal{O}(n) \in \mathcal{A}$ for $n \gg 0$ and therefore all σ -stable factors L_i of $\mathcal{O}(n)$, which by Corollary 2.3 are also spherical, have phases $\phi(L_i) \in (0, 1]$. By (3.2) the L_i are also σ' -stable with σ' -phases $\phi'(L_i) = \phi(L_i)$. Since $\text{Hom}(\mathcal{O}(n), A_m) \neq 0$ implies $\text{Hom}(L_i, A_m) \neq 0$ for at least one L_i , stability yields $0 < \phi(L_i) = \phi'(L_i) \leq \phi'(A_m)$.

Thus one finds $1 > \phi'(A_1) \geq \dots \geq \phi'(A_{m-1}) \geq \phi'(A_m) \geq 0$. Since $Z'(k(x)) = \sum Z'(A_i)$ and $Z'(k(x)) = Z(k(x)) = -1$, this is impossible.

iii) Suppose A_1 is semirigid. Then A_2, \dots, A_m are spherical and hence also σ -stable with phases $\phi(A_i) = \phi'(A_i)$, $i = 2, \dots, m$. Using that $k(x) \rightarrow A_m$ is non-trivial and not an isomorphism, one finds $1 < \phi(A_m) = \phi'(A_m) \leq \dots \leq \phi'(A_2) = \phi(A_2)$. Hence, $A_i \in \mathcal{A}[k_i]$, $i = 2, \dots, m$, with $k_i \geq 1$. Then Lemma 1.4 shows $\text{Hom}(A_i, \mathcal{O}(-n)[2]) = 0$ for $i = 2, \dots, m$ and $n \gg 0$. Since $\text{Hom}(k(x), \mathcal{O}(-n)[2]) = \text{Hom}(\mathcal{O}(-n), k(x))^* \neq 0$ for all n , this yields $\text{Hom}(A_1, \mathcal{O}(-n)[2]) \neq 0$ for $n \gg 0$.

For a fixed such n , consider the σ -stable factors L_i of $\mathcal{O}(-n)[2]$ which are contained in $\mathcal{A}[1]$ and hence $\phi(L_i) \in (1, 2]$. Again, the L_i are spherical (cf. Corollary 2.3) and hence by (3.2) also σ' -stable of phase $\phi'(L_i) = \phi(L_i)$. Since $\text{Hom}(A_1, \mathcal{O}(-n)[2]) \neq 0$ implies $\text{Hom}(A_1, L_i) \neq 0$ for at least one L_i , stability yields $\phi'(A_1) \leq 2$.

Thus one finds $1 < \phi'(A_m) \leq \dots \leq \phi'(A_1) \leq 2$. As above, this contradicts $Z'(k(x)) = Z(k(x)) = -1$.

This concludes the proof of Theorem 3.1 in the case that $\sigma \in V(X)$. In Section 3.3 we will use similar arguments for the case that $\sigma \in \partial V(X)$, but they will have to be applied to small deformations of σ and σ' which makes it more technical.

3.2. Suppose now that $\sigma, \sigma' \in \Sigma$ satisfy $Z = Z'$ and (3.1) (or, equivalently, (3.2)). In order to show that then $\sigma = \sigma'$ it suffices to find an autoequivalence $\Phi \in \text{Aut}(\text{D}^b(X))$ such that $\Phi(\sigma) = \Phi(\sigma')$. Since the set of spherical objects $\mathcal{S} \subset \text{Ob}(\text{D}^b(X))$ is invariant under the action of $\text{Aut}(\text{D}^b(X))$, the new stability conditions $\Phi(\sigma), \Phi(\sigma')$ still satisfy (3.1).

Recall that for any $\sigma \in \Sigma$ there exists $\Phi \in \text{Aut}(\text{D}^b(X))$, such that $\Phi(\sigma)$ is contained in the closure $\overline{U(X)}$ of the open set $U(X) \subset \Sigma$ of all stability conditions with respect to which all point sheaves $k(x)$ are stable of the same phase (see [4]). Moreover, $U(X)$ is a principal $\widetilde{\text{Gl}}^+(2, \mathbb{R})$ -bundle over $V(X) \subset U(X)$ (see [4, Sect. 11] or Section 1.3).

Thus, if Φ can be found such that $\Phi(\sigma) \in U(X)$ (and not only in its closure), then there exists a $g \in \widetilde{\mathrm{Gl}}^+(2, \mathbb{R})$ with $g^{-1}\Phi(\sigma) \in V(X)$. By (3.1), applied to $g^{-1}\Phi(\sigma)$ and $g^{-1}\Phi(\sigma')$, one concludes $g^{-1}\Phi(\sigma) = g^{-1}\Phi(\sigma')$ and hence $\sigma = \sigma'$.

3.3. Eventually we have to deal with the case that one only finds $\Phi \in \mathrm{Aut}(\mathrm{D}^b(X))$ such that $\Phi(\sigma)$ is in the boundary of $U(X)$. By applying an appropriate $g \in \widetilde{\mathrm{Gl}}^+(2, \mathbb{R})$ we can reduce to the case that $\sigma \in \partial V(X)$, i.e. all $k(x)$ are σ -semistable (but not necessarily stable) of phase one, and $\sigma' \in \Sigma$.

Pick a path σ_t , $0 \leq t \ll 1$ with $\sigma_0 = \sigma$ and $\sigma_t \in V(X)$ for $t > 0$. The stability function of σ_t shall be denoted Z_t and for a σ_t -semistable object B its phase is $\phi_t(B)$. Since $Z = Z'$ and $\sigma' \in \Sigma$, the path σ_t (or rather its image in $\mathcal{P}_0^+(X)$) can be lifted uniquely to a path σ'_t in Σ with $\sigma'_0 = \sigma'$. Then, by construction, the stability function Z'_t of σ'_t equals Z_t . The phase of a σ'_t -semistable object B shall be denoted $\phi'_t(B)$.

In the following, σ_t -semistability of an object will mean semistability for all small t (depending on the object) and similarly for σ'_t -semistability. Note that semistability is a closed condition, so semistability for all small $t > 0$ will imply semistability for $t = 0$. The same does not hold for semistability replaced by stability. So, when we say an object is σ_t -stable, it means that it is σ_t -stable for all small $t > 0$. The latter implies that it is also σ -semistable, but not necessarily σ -stable.

We continue to assume (3.1) (or, equivalently, (3.2)) for the two stability conditions σ and σ' . The condition is preserved under small deformation as shown by the following

Lemma 3.3. *Suppose A is a spherical object. Then the path σ_t can be chosen such that A is σ_t -semistable if and only if A is σ'_t -semistable. Moreover, in this case $\phi_t(A) = \phi'_t(A)$.*

Proof. Recall that for fixed $Z \in \mathcal{P}^+(X)$ and an arbitrary norm on $N(X)_{\mathbb{R}}$ there exists a constant C such that for all (-2) -classes $\delta \in N(X)$ one has $\|\delta\|^2 \leq 2(1 + C|Z(\delta)|^2)$. This can be found implicitly in the proof of [4, Lem. 8.1] (and explicitly in the first version of the paper). Hence the set of (-2) -classes $\delta \in N(X)$ with bounded $Z(\delta)$ is finite.

Therefore it suffices to prove that the assertion holds for A once it holds for all spherical objects B with $|Z(B)| < |Z(A)|$. A priori the interval $t \in [0, \varepsilon)$ for which semistability with respect to σ_t resp. σ'_t coincide can get smaller when passing from A to B . But only finitely many steps are necessary and, as we shall see, at each step only finitely many spherical objects are involved.

Suppose A is σ_t -semistable but not σ'_t -semistable for $t > 0$. Then there exists a σ'_t -stable decomposition of A with σ'_t -stable factors B_i such that $\phi'_t(B_1) \geq \dots \geq \phi'_t(B_k)$. The arguments to show this can be found in the proof of [4, Prop. 9.3]. Take a compact neighbourhood K of σ' and consider the set $T(A, K)$ of all objects B that are stable factors of A with respect to some $\sigma'_t \in K$. This set is of bounded mass and by [4, Prop. 9.3] there is a finite chamber structure of K such that for an object $B \in T(A, K)$ (semi)stability is constant within a chamber. This chamber structure can be refined such that within one chamber $\log(\phi'_t(B_1)/\phi'_t(B_2))$ does not change signs for all $B_1, B_2 \in T(A, K)$. By the finiteness of the set of Mukai vectors $\{v(B) \mid B \in T(A, K)\}$ (see [4, Lem. 9.2]) the new chamber structure is still finite. Hence σ' will be in the closure of one chamber and we choose σ'_t in this chamber and find the stable decomposition as claimed.

Note that by the assumption that A is not σ'_t -semistable, one has $\phi'_t(B_1) > \phi'_t(B_k)$ for $t > 0$. Since A is spherical, also its stable factors B_i are spherical (cf. Corollary 2.3). For $t = 0$ one has $\phi(B_1) = \phi'(B_1) = \dots = \phi'(B_k) = \phi(B_k)$, because A is σ' -semistable by (3.1). Hence $Z'(B_i) = Z(B_i) \in Z(A)\mathbb{R}_{>0}$. Since $Z(A) = \sum Z(B_i)$, one has $|Z(B_i)| < |Z(A)|$. But then the assertion of the lemma holds for the B_i which are σ'_t -semistable. (At this point the path σ_t has to be adjusted to work for the B_i as well. As mentioned earlier, this procedure really works, because only finitely many objects are eventually used.) Thus, the B_i are σ_t -semistable with $\phi_t(B_i) = \phi'_t(B_i)$. Hence $\phi_t(B_1) = \phi'_t(B_1) > \phi'_t(B_k) = \phi_t(B_k)$ for $t > 0$ contradicting the σ_t -semistability of A .

If A is semistable with respect to σ_t and σ'_t , then $\phi(A) = \phi'(A)$ by (3.1). As $Z_t = Z'_t$, this yields $\phi_t(A) = \phi'_t(A)$. \square

Let us now turn to the proof of Theorem 3.1 in this situation. Morally, Lemma 3.3 says that we can apply Section 3.1 to the stability conditions σ_t and σ'_t for some small $t > 0$. However, the chamber structure that takes care of all the objects involved may not be locally finite near σ' . Indeed, one would start with the σ'_t -stable factors A_i of some $k(x)$ and in the next step would need to consider the σ_t -stable factors of the A_i and so forth. So we have to run the arguments of Section 3.1 once more while keeping track of the deformation to the interior of $V(X)$ (which makes everything more technical).

We shall prove that each $k(x)$ is σ'_t -stable of phase one for small $t > 0$. Since the family of all $k(x)$ is of bounded mass in Σ , this suffices to conclude that $\sigma' \in \partial V(X)$. Indeed by [4, Prop. 9.3] the chamber structure of a compact neighbourhood of σ' with respect to $\{k(x)\}$ is finite and hence all $k(x)$ will be σ'_t -semistable for t small but independent of the particular point sheaf $k(x)$. Moreover, as in Section 3.1, the phase will be one and hence σ'_t is a standard stability condition. Then $Z'_t = Z_t$ and the fact that a standard stability condition is determined by its stability function shows $\sigma_t = \sigma'_t$ and hence $\sigma = \sigma'$.

Suppose $k(x)$ is not σ'_t -stable. Then there exists a decomposition as in Section 3.1 with factors A_1, \dots, A_m which are σ'_t -stable and satisfy $\phi'_t(A_1) \geq \dots \geq \phi'_t(A_m)$. This follows from [4, Prop. 9.3] (see also the arguments in the proof of Lemma 3.3). Note that then A_1, \dots, A_m are still σ' -semistable but not necessarily σ' -stable. In the following, we use similar arguments as in Section 3.1. In particular, we distinguish three cases.

i) Suppose A_1 and A_m are both spherical. Then by (3.1) they are also σ -semistable with $\phi(A_1) = \phi'(A_1)$ and $\phi(A_m) = \phi'(A_m)$. Due to the existence of the non-trivial morphisms $A_1 \rightarrow k(x)$ and $k(x) \rightarrow A_m$ and the σ -semistability of $k(x)$, this yields $\phi'(A_1) = \phi(A_1) \leq \phi(k(x)) = 1$ and $1 = \phi(k(x)) \leq \phi(A_m) = \phi'(A_m)$. Together with $\phi'_t(A_1) \geq \dots \geq \phi'_t(A_m)$ one finds that $k(x)$ is σ' -semistable. In fact more is true. Since $k(x)$ is σ_t -stable for $t > 0$ and by Lemma 3.3 A_1 and A_m are σ_t -semistable with $\phi_t(A_i) = \phi'_t(A_i)$, one obtains $1 \geq \phi_t(A_1) \geq \phi_t(A_m) \geq 1$. As the σ_t -stable semirigid $k(x)$ cannot be a σ_t -stable factor of the spherical A_1 (use Corollary 2.3), the first inequality must be strict which is absurd for $m > 1$. Thus, $k(x)$ is σ'_t -stable for $t > 0$ of phase one. Hence, if we are in case **i)** for all $x \in X$, then $\sigma_t \in \partial V(X)$.

ii) Suppose A_m is semirigid. Then A_1, \dots, A_{m-1} are spherical and by (3.1) also σ -semistable of phase $\phi(A_i) = \phi'(A_i)$. The existence of the non-trivial $A_1 \twoheadrightarrow k(x)$ and σ -semistability of $k(x)$ yield $1 \geq \phi(A_1) = \phi'(A_1) \geq \dots \geq \phi'(A_{m-1}) = \phi(A_{m-1})$.

By Lemma 3.3 the A_i , $i = 1, \dots, m-1$, are σ_t -semistable of phase $\phi_t(A_i) = \phi'_t(A_i)$. Thus, $\phi_t(A_1) = \phi'_t(A_1) \geq \dots \geq \phi'_t(A_{m-1}) = \phi_t(A_{m-1})$. Moreover, $\sigma_{t>0}$ -stability of $k(x)$ implies $1 = \phi_t(k(x)) \geq \phi_t(A_1)$ for $t > 0$. (Actually, $\phi_t(k(x)) = \phi_t(A_1)$ can be excluded for $t > 0$, because as above the semirigid $k(x)$ cannot be a stable factor of the spherical A_1 , see Corollary 2.3). Thus A_1, \dots, A_{m-1} are $\sigma_{t>0}$ -semistable of phase ≤ 1 (in fact, < 1) and by Lemma 2.6 this proves $\text{Hom}(\mathcal{O}(n), A_i) = 0$ for $n \gg 0$ and $i = 1, \dots, m-1$. As in Section 3.1, ii) this yields $\text{Hom}(\mathcal{O}(n), A_m) \neq 0$ for $n \gg 0$.

Let now L_1, \dots, L_k be the σ -stable factors of $\mathcal{O}(n)$ with $\phi(L_1) \geq \dots \geq \phi(L_k)$. They are again spherical and hence by (3.1) also σ' -stable. Stability is an open property for objects with primitive Mukai vector (see [4, Prop. 9.4]). Hence the L_i are stable with respect to σ_t and σ'_t and, moreover, $\phi_t(L_i) = \phi'_t(L_i)$. For $t > 0$ all stable factors of $\mathcal{O}(n)$, $n \gg 0$, have phases in $(0, 1]$. The non-trivial $\mathcal{O}(n) \twoheadrightarrow L_k$ and σ_t -stability of L_k therefore imply $\phi_t(L_k) > 0$ for $t > 0$. In the limit we still have $\phi(L_k) \geq 0$ and hence $\phi(L_i) \geq 0$ for all L_i .

Since $\text{Hom}(L_i, A_m) \neq 0$ for at least one L_i , semistability of A_m and L_i with respect to σ' yields $\phi'(L_i) \leq \phi'(A_m)$ and hence $0 \leq \phi'(A_m) \leq \dots \leq \phi'(A_1) \leq 1$. (The last inequality is a priori not strict.) This contradicts $Z'(k(x)) = -1$ except for the case that $\phi'(A_m) = \dots = \phi'(A_1) = 1$. However, if $\phi'(A_m) = 1$, then for small $t > 0$ one still has $\phi'_t(A_m) > 0$ and thus $0 < \phi'_t(A_m) \leq \dots \leq \phi'_t(A_1) < 1$ where the last inequality is strict for $t > 0$. This contradicts $Z'_t(k(x)) = -1$.

iii) Suppose A_1 is semirigid. Then A_2, \dots, A_m are spherical and hence by (3.1) also σ -semistable of phase $\phi(A_i) = \phi'(A_i)$. The existence of the non-trivial $k(x) \twoheadrightarrow A_m$ implies $\phi(A_2) = \phi'(A_2) \geq \dots \geq \phi'(A_m) = \phi(A_m) \geq \phi(k(x)) = 1$. By Lemma 3.3 we know that A_2, \dots, A_m are σ_t -semistable of phase $\phi_t(A_i) = \phi'_t(A_i)$ and thus $\phi_t(A_2) = \phi'_t(A_2) \geq \dots \geq \phi'_t(A_m) = \phi_t(A_m) > \phi(k(x)) = 1$. The last inequality is strict because the $\sigma_{t>0}$ -stable semirigid $k(x)$ cannot be a stable factor of the spherical A_m (Corollary 2.3).

By Lemma 1.4 one then has $\text{Hom}(A_i, \mathcal{O}(-n)[2]) = 0$ for $n \gg 0$ and $i = 2, \dots, m$. And as in Section 3.1, iii) this yields $\text{Hom}(A_1, \mathcal{O}(-n)[2]) \neq 0$. Consider the σ -stable factors L_1, \dots, L_k of $\mathcal{O}(-n)[2]$ with $\phi(L_1) \geq \dots \geq \phi(L_k)$. Since they are spherical, they are also σ' -stable and hence σ'_t -stable, as stability is open for objects with primitive Mukai vector by [4, Prop. 9.4]. Using Lemma 3.3 one finds that the L_i are semistable with respect to σ'_t and σ_t . Moreover, $\phi'_t(L_i) = \phi_t(L_i)$. Using that all σ_t -stable factors of $\mathcal{O}(-n)[2]$ have phases in $(1, 2]$ and the existence of the non-trivial $L_1 \twoheadrightarrow \mathcal{O}(-n)[2]$, one finds $\phi_t(L_1) \leq 2$. Thus also $\phi(L_1) \leq 2$ and hence $\phi(L_k) \leq \dots \leq \phi(L_1) \leq 2$.

As $\text{Hom}(A_1, L_i) \neq 0$ for at least one L_i and both, A_1 and L_i , are σ' -stable, one finds $1 \leq \phi'(A_m) \leq \dots \leq \phi'(A_1) \leq 2$. The latter contradicts $Z'(k(x)) = -1$ except for the case that $\phi'(A_m) = \dots = \phi'(A_1) = 1$. However, if $\phi'(A_1) = 1$, then for small $t > 0$ still $\phi'_t(A_1) < 2$ and hence $1 < \phi'_t(A_m) \leq \dots \leq \phi'_t(A_1) < 2$ where the first inequality is strict for $t > 0$. This, once more, contradicts $Z'_t(k(x)) = -1$.

Remark 3.4. The rough idea of the above arguments goes as follows. If for two stability conditions σ and σ' with the same stability function $Z = Z'$ an object E is stable with respect to σ but not with respect to σ' , then pass to the σ' -stable factors A_i of E . Either for all A_i one has $\text{ext}^1(A_i, A_i) < \text{ext}^1(E, E)$ or all A_i are spherical except for one, say A_{i_0} , for which $\text{ext}^1(A_{i_0}, A_{i_0}) = \text{ext}^1(E, E)$. By induction hypothesis one can assume that σ' -stable A with $\text{ext}^1(A, A) < \text{ext}^1(E, E)$ are σ -stable of the same phase. So the difficult case is the one that E has a σ' -stable factor with the same ext^1 and one needs to derive a contradiction here, somehow.

4. THE SPHERICAL METRIC

4.1. We shall define a ‘spherical’ version of Bridgeland’s metric (see Section 1.2). Instead of testing all object in \mathcal{D} only spherical objects are taken into account.

We start out with the space of slicings. As before, \mathcal{S} denotes the set of spherical objects in a K3 category \mathcal{D} .

Definition 4.1. For two slicings \mathcal{P} and \mathcal{P}' one defines

$$f_{\mathcal{S}}(\mathcal{P}, \mathcal{P}') := \sup_{0 \neq A \in \mathcal{S}} \{|\phi^+(A) - \phi'^+(A)|, |\phi^-(A) - \phi'^-(A)|\}.$$

Clearly, $f_{\mathcal{S}}(\mathcal{P}, \mathcal{P}') \leq f(\mathcal{P}, \mathcal{P}')$ (see Section 1.2 for the definition of f) and thus the standard topology is a priori finer than the one defined by $f_{\mathcal{S}}$.

Remark 4.2. i) Note that $f_{\mathcal{S}}$ for a general K3 category \mathcal{D} will usually not be a generalized metric and possibly not even well-defined. Eg. if \mathcal{D} has no or too few spherical objects, then $f_{\mathcal{S}}$ is not defined (although one could set it constant zero in this case) or one could have $f_{\mathcal{S}}(\mathcal{P}, \mathcal{P}') = 0$ without $\mathcal{P} = \mathcal{P}'$.

ii) Note that $f_{\mathcal{S}}(\mathcal{P}, \mathcal{P}') = 0$ if and only if $\mathcal{P}(\phi) \cap \mathcal{S} = \mathcal{P}'(\phi) \cap \mathcal{S}$ for all ϕ . The ‘only if’ is obvious. For the other direction, consider $A \in \mathcal{S}$ with \mathcal{P} -stable factors A_1, \dots, A_n of phase $\phi_1 \geq \dots \geq \phi_n$, which are spherical by Corollary 2.3. If $\mathcal{P}(\phi) \cap \mathcal{S} = \mathcal{P}'(\phi) \cap \mathcal{S}$ for all ϕ , then $A_i \in \mathcal{P}'(\phi_i)$ and hence $\phi^{\pm}(A) = \phi'^{\pm}(A)$. This proves $f_{\mathcal{S}}(\mathcal{P}, \mathcal{P}') = 0$.

Note that by Proposition 2.5 $f_{\mathcal{S}}(\mathcal{P}, \mathcal{P}') = 0$ is also equivalent to the condition $\mathcal{P}(\phi)^s \cap \mathcal{S} = \mathcal{P}'(\phi)^s \cap \mathcal{S}$ for all ϕ .

Consider two stability conditions $\sigma = (\mathcal{P}, Z), \sigma' = (\mathcal{P}', Z')$ on \mathcal{D} .

Definition 4.3. The *spherical metric* $d_{\mathcal{S}}(\sigma, \sigma')$ is defined as

$$d_{\mathcal{S}}(\sigma, \sigma') := \max\{f_{\mathcal{S}}(\mathcal{P}, \mathcal{P}'), |Z - Z'|\}.$$

4.2. Consider a complex projective K3 surface X and let Σ be Bridgeland’s distinguished connected component of the space $\text{Stab}(X)$ of locally finite numerical stability conditions on $\text{D}^b(X)$ (see Section 1.3). Let $\sigma = (\mathcal{P}, Z)$ and $\sigma' = (\mathcal{P}', Z')$ be stability conditions contained in Σ . The following is the analogue of [3, Lem. 6.4].

Proposition 4.4. i) If $d_{\mathcal{S}}(\sigma, \sigma') = 0$, then $\sigma = \sigma'$.

ii) If $Z = Z'$ and $f_{\mathcal{S}}(\mathcal{P}, \mathcal{P}') < 1$, then $\sigma = \sigma'$.

Proof. Let us first prove that i) implies ii). Here we adapt the original arguments in [3, Lem. 6.4], avoiding non-spherical objects. So we have to show that $Z = Z'$ and $f_{\mathcal{S}}(\mathcal{P}, \mathcal{P}') < 1$ imply $f_{\mathcal{S}}(\mathcal{P}, \mathcal{P}') = 0$, i.e. that for all ϕ one has $\mathcal{P}(\phi) \cap \mathcal{S} = \mathcal{P}'(\phi) \cap \mathcal{S}$. Assuming $A \in \mathcal{P}(\phi) \cap \mathcal{S}$, we have to prove $A \in \mathcal{P}'(\phi)$. If $A \in \mathcal{P}'(> \phi)$, then from $f_{\mathcal{S}}(\mathcal{P}, \mathcal{P}') < 1$ one deduces $A \in \mathcal{P}'(\phi, \phi + 1)$. The latter would contradict $Z(A) = Z'(A)$. Similarly one excludes the case $A \in \mathcal{P}'(< \phi)$.

Consider the σ' -stable factors A_1, \dots, A_k of A with $\phi'_1 := \phi'(A_1) \geq \dots \geq \phi'_k := \phi'(A_k)$. Note that A_1, \dots, A_k are again spherical and that $\phi + 1 > \phi'_1 \geq \dots \geq \phi'_k > \phi - 1$. We have dealt already with the cases that $\phi > \phi'_1$ or $\phi'_k > \phi$. So we may assume $\phi'_1 \geq \phi \geq \phi'_k$ and have to show equality. If exactly one of the inequalities is strict, then $Z = Z'$ yields a contradiction. So we may assume $\phi + 1 > \phi'_1 \geq \dots \geq \phi'_\ell > \phi \geq \phi'_{\ell+1} \geq \dots \geq \phi'_k > \phi - 1$ for some $1 \leq \ell \leq k$.

Breaking the filtration at the ℓ -th step yields an exact triangle $B_1 \rightarrow A \rightarrow B_2$, i.e. the σ' -stable factors of B_1 are A_1, \dots, A_ℓ and the σ' -stable factors of B_2 are $A_{\ell+1}, \dots, A_k$.

One now proves that $B_1 \in \mathcal{P}(> \phi - 1)$ and $B_2 \in \mathcal{P}(\leq \phi + 1)$. If $B_1 \notin \mathcal{P}(> \phi - 1)$, then there exists a σ -stable object C of phase $\phi(C) \leq \phi - 1$ such that $\text{Hom}(B_1, C) \neq 0$. Then for at least one of the σ' -stable factors A_1, \dots, A_ℓ of B_1 , say A_{i_0} , one has $\text{Hom}(A_{i_0}, C) \neq 0$ and, in a next step, for one σ -stable factor A'_{i_0} of A_{i_0} one has $\text{Hom}(A'_{i_0}, C) \neq 0$. Since the spherical A_{i_0} is σ' -stable of phase $\phi'_{i_0} > \phi$, its σ -stable factor A'_{i_0} has phase $\phi(A'_{i_0}) \in (\phi'_i - 1, \phi'_{i_0} + 1)$. Hence $\phi(A'_{i_0}) > \phi - 1$, which contradicts $\text{Hom}(A'_{i_0}, C) \neq 0$. The argument to prove $B_2 \in \mathcal{P}(\leq \phi + 1)$ is similar.

Next one shows that $B_1 \in \mathcal{P}(> \phi - 1)$ excludes $B_1 \in \mathcal{P}(\leq \phi)$. Indeed, otherwise $B_1 \in \mathcal{P}(\phi - 1, \phi]$ and hence $Z(B_1) \in \exp(i\pi\varphi)\mathbb{R}_{>0}$ for some $\varphi \in (\phi - 1, \phi]$. But $Z(B_1) = Z'(B_1) = \sum_{i=1}^{\ell} Z'(A_i) \in \sum_{i=1}^{\ell} \exp(i\pi\phi'_i)\mathbb{R}_{>0}$ with $\phi'_i \in (\phi, \phi + 1)$. Contradiction.

Now one concludes as in [3]. As $B_1 \notin \mathcal{P}(\leq \phi)$, there exists a σ -stable object C of phase $\phi(C) > \phi$ with $\text{Hom}(C, B_1) = 0$. Since $A \in \mathcal{P}(\phi)$ one has $\text{Hom}(C, A) = 0$ and hence $\text{Hom}(C, B_2[-1]) \neq 0$. But the latter is excluded due to $B_2[-1] \in \mathcal{P}(\leq \phi)$.

To prove i) one observes that $d_{\mathcal{S}}(\sigma, \sigma') = 0$ implies $Z = Z'$ and $f_{\mathcal{S}}(\mathcal{P}, \mathcal{P}') = 0$ and that by Remark 4.2 the latter is equivalent to $\mathcal{P}(\phi) \cap \mathcal{S} = \mathcal{P}'(\phi) \cap \mathcal{S}$ for all $\phi \in \mathbb{R}$. From Theorem 3.1 one concludes $\sigma = \sigma'$. \square

Corollary 4.5. *The classical metric d and the spherical metric $d_{\mathcal{S}}$ define equivalent topologies on Σ .*

Proof. By Proposition 4.4, the projection $\Sigma \rightarrow \mathcal{P}_0^+(X)$ is a local homeomorphism also for the topology induced by $d_{\mathcal{S}}$. \square

4.3. Stability conditions on spherical collections. Ideally, we would like to talk about stability conditions on the set \mathcal{S} , possibly viewed with its structure as a \mathbb{C} -linear category or with the binary operation $(A, B) \mapsto T_A(B)$ induced by spherical twists. However, there does not seem a way around the σ -stable filtrations and, although all the stable factors A_i in (1.1) for $E \in \mathcal{S}$ are spherical (and for the spherical metric one only needs A_1 and A_m), the filtrations as such are not intrinsic to \mathcal{S} .

So instead we consider $\mathcal{S}^* \subset \text{D}^b(X)$, the smallest full triangulated subcategory containing \mathcal{S} . In other words \mathcal{S} generates \mathcal{S}^* without taking direct summands. For details see [10]. As noted there, \mathcal{S}^* is a triangulated category with a reasonably small Grothendieck group

$K(\mathcal{S}^*) = N(X) \subset H^*(X, \mathbb{Z})$ (assuming $\rho(X) \geq 2$), but without bounded t-structures (we are working over \mathbb{C} !). As explained in [10], the category \mathcal{S}^* is expected to be $D^b(X)$ for K3 surfaces over $\bar{\mathbb{Q}}$.

Remark 4.6. Note in passing that the numerical Grothendieck group of \mathcal{S}^* , i.e. $K(\mathcal{S}^*) := K(\mathcal{S}^*)/\sim_{\mathcal{S}}$, equals $K(X)/\sim = N(X)$, for the Mukai pairing is non-degenerated on $N(X)$.

So even passing to \mathcal{S}^* will not allow us to speak about stability conditions on \mathcal{S} or, rather, on \mathcal{S}^* . For this reason we allow ourselves to adapt the original notion as follows. Let \mathcal{T} be a K3 category and $\mathcal{S} \subset \mathcal{T}$ a generating collection of spherical objects invariant under shift. For our purposes take $\mathcal{T} = \mathcal{S}^*$.

Definition 4.7. A *stability condition* σ on \mathcal{S} with respect to $\mathcal{S} \subset \mathcal{T}$ consists of an additive stability function $Z : K(\mathcal{T})/\sim \rightarrow \mathbb{C}$ and subsets $\mathcal{S}(\phi) \subset \mathcal{S}$, $\phi \in \mathbb{R}$ satisfying the following conditions: i) $\mathcal{S}(\phi)[1] = \mathcal{S}(\phi + 1)$, ii) $\text{Hom}(\mathcal{S}(\phi_1), \mathcal{S}(\phi_2)) = 0$ for $\phi_1 > \phi_2$, iii) $Z(\mathcal{S}(\phi)) \subset \exp(i\pi\phi)\mathbb{R}_{>0}$, and iv) for every $E \in \mathcal{S}$ there exists a filtration as in (1.1) with $A_i \in \mathcal{S}(\phi_i)$.

Remark 4.8. In order to think of this notion as a stability condition on \mathcal{S} , i.e. independent of \mathcal{T} , one would need some kind of ‘formality’ statement saying that \mathcal{T} is uniquely determined by the \mathbb{C} -linear category \mathcal{S} . For certain ‘spherical configurations’ this is indeed true (cf. [12, 14, 15]). In our context one would in particular have to decide whether any \mathbb{C} -linear equivalence $\mathcal{S}_{X_1} \simeq \mathcal{S}_{X_2}$ for the spherical collections $\mathcal{S}_{X_i} \subset D^b(X_i)$, $i = 1, 2$, of two K3 surfaces X_1, X_2 always extends to an exact equivalence $D^b(X_1) \simeq D^b(X_2)$ (see [10]).

Note that for a generic non-projective K3 surface \mathcal{S} consists of shifts of \mathcal{O}_X (see [7]). In this case, \mathcal{S}^* is the unique K3 category generated by a spherical object (cf. [12]).

Let $\text{Stab}(\mathcal{S}) := \text{Stab}(\mathcal{S} \subset \mathcal{T})$ be the space of stability conditions on \mathcal{S} with respect to $\mathcal{S} \subset \mathcal{T}$ in the sense of Definition 4.7. It can be equipped with a generalized metric $d_{\mathcal{S}}$ as in Definition 4.3. We do not intend to develop the theory here fully, but most of the arguments in [3] can be adapted. A good example is maybe Proposition 4.4, which works in this setting.

In any case, it is obvious that for $\mathcal{T} = \mathcal{S}^* \subset D^b(X)$ the restriction of a stability condition on $D^b(X)$ to \mathcal{S}^* yields a stability condition on \mathcal{S} (with respect to $\mathcal{S} \subset \mathcal{S}^*$) in the above sense. The induced map

$$\text{Stab}(X) \longrightarrow \text{Stab}(\mathcal{S})$$

is continuous with respect to the corresponding metrics. The pull-back of the metric on $\text{Stab}(\mathcal{S})$ yields the spherical metric $d_{\mathcal{S}}$ on $\text{Stab}(X)$. The main result can thus be reformulated as

Corollary 4.9. *On the distinguished component the restriction yields an embedding*

$$\Sigma \hookrightarrow \text{Stab}(\mathcal{S})$$

which identifies the natural metric on $\text{Stab}(\mathcal{S})$ with the spherical metric $d_{\mathcal{S}}$ on Σ . □

APPENDIX A. GROUP OF AUTOEQUIVALENCES

The following remarks are largely independent of the rest of the paper, but can be seen as a motivation for the study of $\text{Stab}(X) \longrightarrow \text{Stab}(\mathcal{S})$.

A.1. We shall first fix (or recall) some notations. As before, X denotes a complex projective K3 surface and $D^b(X)$ its bounded derived category of coherent sheaves. Due to the Global Torelli theorem the group $\text{Aut}(X)$ of automorphisms of X can be identified with a subgroup of all Hodge isometries of $H^2(X, \mathbb{Z})$. We will write this as

$$\text{Aut}(X) \hookrightarrow \text{O}(H^2(X, \mathbb{Z})).$$

The group of *transcendental automorphisms* $\text{Aut}^t(X)$ is by definition the subgroup of $\text{Aut}(X)$ consisting of all $f \in \text{Aut}(X)$ for which the induced action $f^* \in \text{O}(H^2(X, \mathbb{Z}))$ is trivial on the algebraic part $\text{NS}(X)$. Thus,

$$\text{Aut}^t(X) \hookrightarrow \text{O}(T(X)),$$

where $T(X) \subset H^2(X, \mathbb{Z})$ is the transcendental lattice. It is known that $\text{Aut}^t(X)$ is a finite group.

The group of linear exact autoequivalences of $D^b(X)$ is denoted $\text{Aut}(D^b(X))$. It comes with a natural representation

$$\rho : \text{Aut}(D^b(X)) \longrightarrow \text{O}(\tilde{H}(X, \mathbb{Z})),$$

which is not injective and we shall denote its kernel by $\text{Aut}_0(D^b(X))$. The description of the image of ρ was completed in [8]. Since any autoequivalence naturally acts on $\text{Stab}(X)$, one can define the two subgroups

$$\text{Aut}^\Sigma(D^b(X)) \subset \text{Aut}(D^b(X)) \text{ and } \text{Aut}_0^\Sigma(D^b(X)) \subset \text{Aut}_0(D^b(X))$$

of autoequivalences that respect the distinguished component $\Sigma \subset \text{Stab}(X)$. Conjecturally, one has $\Sigma = \text{Stab}(X)$ or, less optimistic, $\text{Aut}^\Sigma(D^b(X)) = \text{Aut}(D^b(X))$.

A.2. Instead of letting an automorphism of X or an autoequivalence of $D^b(X)$ act on the cohomology, we can study its action on the collection of spherical objects $\mathcal{S} \subset \text{Ob}(D^b(X))$. We shall denote these *spherical actions* by

$$\tau : \text{Aut}(X) \longrightarrow \text{Aut}(\mathcal{S}) \text{ and } \tau : \text{Aut}(D^b(X)) \longrightarrow \text{Aut}(\mathcal{S}).$$

Note that the set $v(\mathcal{S}) \subset \tilde{H}(X, \mathbb{Z})$ of Mukai vectors of all spherical objects generates the algebraic part $N(X)$ of $\tilde{H}(X, \mathbb{Z})$. This immediately shows

$$(A.1) \quad \ker(\tau : \text{Aut}(X) \longrightarrow \text{Aut}(\mathcal{S})) \subset \text{Aut}^t(X).$$

Remark A.1. Presumably equality holds in (A.1), but the only thing that seems obvious is the following. Let $f \in \text{Aut}^t(X)$ and suppose $A \in \mathcal{S}$ is a spherical object that is stable with respect to some stability condition $\sigma \in \Sigma$. Then $f^*A \simeq A$.

For spherical sheaves which are μ -stable with respect to some ample line bundle H on X this is due to a well-known argument of Mukai. If A is μ_H -stable, then f^*A is μ_{f^*H} -stable. Since $f^*H = H$ for a transcendental f , both sheaves A and $A' = f^*A$ are μ_H -stable. Moreover, they have the same Mukai vector and hence $\chi(A, A') = 2$ which shows that there must exist a non-trivial homomorphism between A and A' . The latter together with the stability of the two sheaves yields $A \simeq A' = f^*A$.

For the general case of a spherical object A that is stable with respect to some $\sigma \in \Sigma$, one uses that $f^*\sigma = \sigma$ (see proof of Lemma A.3) and argues as above. One could try to deal with

an arbitrary spherical object by applying the above to its stable factors (with respect to some $\sigma \in \Sigma$), but due to the non-trivial action of f on the Hom-spaces this is not obvious.

Lemma A.2. *For the spherical representations one has*

$$\ker(\tau : \operatorname{Aut}(\operatorname{D}^b(X)) \longrightarrow \operatorname{Aut}(\mathcal{S})) = \ker(\tau : \operatorname{Aut}(X) \longrightarrow \operatorname{Aut}(\mathcal{S})) \subset \operatorname{Aut}^t(X).$$

Proof. Let $\Phi \in \operatorname{Aut}(\operatorname{D}^b(X))$ act trivially on \mathcal{S} . In particular, Φ leaves invariant powers L^i of an ample line bundle L . The action on the graded ring $\bigoplus H^0(X, L^i)$ is induced by an automorphism $f \in \operatorname{Aut}(X, L)$. Hence Φ and f^* define two autoequivalences which are isomorphic on the full subcategory given by the ample sequence $\{L^i\}$. By a result of Bondal and Orlov (see [1] or [6, Prop. 4.23]), this immediately yields $\Phi = f^*$. But then $f \in \operatorname{Aut}^t(X)$. \square

A.3. The groups $\operatorname{Aut}(X)$ and $\operatorname{Aut}(\operatorname{D}^b(X))$ both act on $\operatorname{Stab}(X)$. We denote this action by

$$\kappa : \operatorname{Aut}(\operatorname{D}^b(X)) \longrightarrow \operatorname{Aut}(\operatorname{Stab}(X)).$$

The main result of [4] says that the subgroup $\operatorname{Aut}_0^\Sigma(\operatorname{D}^b(X))$ is via κ identified with the group of deck transformations of $\Sigma \longrightarrow \mathcal{P}_0^+(X)$:

$$\kappa : \operatorname{Aut}_0^\Sigma(X) \xrightarrow{\sim} \operatorname{Gal}(\Sigma/\mathcal{P}_0^+(X)).$$

Lemma A.3. *For the action on the space of stability conditions one has*

$$\ker(\kappa : \operatorname{Aut}^\Sigma(\operatorname{D}^b(X)) \longrightarrow \operatorname{Aut}(\Sigma)) = \ker(\kappa : \operatorname{Aut}(X) \longrightarrow \operatorname{Aut}(\Sigma)) = \operatorname{Aut}^t(X).$$

Proof. Let first $f \in \operatorname{Aut}(X)$ and consider a standard stability condition σ with stability function $Z(E) = \langle \exp(B + i\omega), v(E) \rangle$. Then $f^*\sigma$ is a stability function for which by definition all $f^*k(x)$ are stable. Hence all point sheaves $k(y)$ are again stable and for $f \in \operatorname{Aut}^t(X)$ the stability function remains unchanged under pull-back. Thus, for standard stability conditions σ one has $f^*\sigma = \sigma$. In particular, f^* preserves the distinguished component Σ and acts on it as a deck-transformation with fixed points. Hence $f^* = \operatorname{id}$ on Σ .

Conversely, if Φ acts trivially on Σ , then Φ acts trivially on $\operatorname{NS}(X)$. By the Global Torelli theorem the induced action on $T(X)$ is of the form f^* for some $f \in \operatorname{Aut}^t(X)$. Changing Φ by the inverse of f^* , we may assume that Φ acts trivially on $\tilde{H}(X, \mathbb{Z})$. But then $\Phi \in \ker(\kappa : \operatorname{Aut}_0^\Sigma(\operatorname{D}^b(X)) \longrightarrow \operatorname{Gal}(\Sigma/\mathcal{P}_0^+(X)))$ and thus $\Phi = \operatorname{id}$. \square

The observation that the kernels of the two actions

$$\tau : \operatorname{Aut}(\operatorname{D}^b(X)) \longrightarrow \operatorname{Aut}(\mathcal{S}) \quad \text{and} \quad \kappa : \operatorname{Aut}(\operatorname{D}^b(X)) \longrightarrow \operatorname{Aut}(\Sigma)$$

essentially coincide hints at the deeper that stability conditions in Σ are determined by their behavior with respect to \mathcal{S} which is expressed by Theorem 3.1 and Corollary 4.9.

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